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Introduction

Denote V the set of all real numbers, M the multiplicative free V-module of rank n, consisting of all words of the form $a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n}$ over the alphabet $a_1, a_2, \dots, a_n$ where α_r $(r = 1, 2, \dots, n)$ runs over the elements of the set V. The identity element of the module M will be denoted by ε.

Let us denote by F^+ the set of all elements of M having the form $a_{i_1}^{x_1} a_{i_2}^{x_2} \dots a_{i_r}^{x_r}$, $(x_i > 0$, $1 \leqq r \leqq n)$ together with the identity element of M. The set of inverses of elements in F^+ will be denoted by F^-. The couple a, b of elements of F^+ will be called relatively prime if the only common factor of a and b in F^+ is ε.

If, for elements a, b in F^+ we have $a = a'z$ and $b = b'z$, $(a', b', z \in F^+)$, furthermore a' and b' are relatively prime then the element z is called the "greatest common divisor" of a and b and is denoted by $a \Delta b$. The „least common multiple" of elements a, b in F^+ is defined by $ab(a\Delta b)^{-1}$, and is denoted by $a \nabla b$. We shall define an operation: $\otimes$ on the set $F^+ \times F^+$ as follows: The $\otimes$ product of the elements (a, b), (c, d) is defined if and only if $b \Delta c \neq \varepsilon$, and in this case $(a, b) \otimes (c, d) = (ac(b\Delta c)^{-1}, bd(b\Delta c)^{-1})$.

It can be shown that for the elements (a, b), (c, d), (g, h) in $F^+ \times F^+$ we have $\big((a, b) \otimes (c, d)\big) \otimes (g, h) = (a, b) \otimes \big((c, d) \otimes (g, h)\big)$ provided that $b \Delta c \neq \varepsilon$, $g \Delta d \neq \varepsilon$.

In this lecture we shall investigate the structure, and special ideals of the partial semigroup $P(F^+ \times F^+, \otimes)$.

The results of this work play an important role in the theory of systems of chemical engineering. For the study of the field treated in this lecture, the reader is refferd to the works [1], [2], [3].

1. The basic properties of the partial semigroup $P(F^+ \times F^+, \otimes)$

Denote L(a, b) resp. R(a, b) the set of all elements (x,y) from $F^+ \times F^+$ for which the product $(x,y) \otimes (a, b)$ resp. $(a, b) \otimes (x,y)$ does exist.

Denote H(a) the set of all elements of F^+ for which $a \Delta z \neq \varepsilon$.

It can be seen at once that the structure of L(a, b), [R(a, b)] is essentially determined by that of H(a), [H(b)] in view of the relations

$$L(a,b) = \bigcup_{x \in F^+} \bigcup_{y \in H(a)} (x,y)$$

resp.
$$R(a,b)=\bigcup_{y\in F^+}\bigcup_{x\in H(b)}(x,y).$$

In the case of $a\in F^+$, $x,y,z\in H(a)$ one has $a\Delta(xy)\neq\varepsilon$, and $(xy)z=x(yz)$, hence it follows that $H(a)$ is a subsemigroup of F^+, furthermore $\varepsilon\notin H(a)$ because of $a\Delta\varepsilon=\varepsilon$.

The set $F^+\backslash\overline{H}(a)=H(a)$ is a submonoid of F^+ because $\varepsilon\in F^+\backslash H(a)$ and $x,y,z\in F^+\backslash H(a)$ imply $a\Delta x=\varepsilon$, $b\Delta x=\varepsilon$ and $a\Delta(xy)=\varepsilon$ whence $xy\in F^+\backslash H(a)$ moreover $(xy)z=x(yz)$ in the semigroup F^+.

Theorem 1.1. If $a,b\in F^+$, then $\overline{H}(a)\cap\overline{H}(b)=\overline{H}(ab)$.

Proof. If $x\in\overline{H}(ab)$, then $(ab)\Delta x=\varepsilon$, $a\Delta x=\varepsilon$ and $x\Delta(ab)=\varepsilon$, whence $x\in\overline{H}(ab)$ and the inclusion $\overline{H}(a)\cap\overline{H}(b)\subseteq\overline{H}(ab)$ follows.

Hence in view of the converse inclusion

$\overline{H}(ab)\subseteq\overline{H}(a)\cap\overline{H}(b)$ indeed $\overline{H}(a)\cap\overline{H}(b)=\overline{H}(ab)$.

Hence for any element a of F^+ we have

$$\overline{H}(a^n)=\overline{H}(a),\ (n=1,2,3,\ldots).$$

Theorem 1.2. If $a,b\in F^+$ we have

$$H(ab)=H(a)\cup H(b).$$

Proof. Since in case of $a,b,x\in F^+$, $x\Delta a\neq\varepsilon$, it holds $x\Delta ab\neq\varepsilon$, whence $H(a)\subseteq H(ab)$. Thus indeed $H(a)\cup H(b)\subseteq H(ab)$, because of $H(a)\subseteq H(ab)$, $H(b)\subseteq H(ab)$.

If $x\in H(ab)$ then $x\Delta ab\neq\varepsilon$ and hence at least one of the relations $x\Delta a\neq\varepsilon$, $x\Delta b\neq\varepsilon$ holds.

Therefore $x\in H(a)\cup H(b)$ whence $H(ab)\subseteq H(a)\cup H(b)$. The inclusions $H(a)\cup H(b)\subseteq$ $\subseteq H(ab)$, $H(ab)\subseteq H(a)\cup H(b)$ imply

$$H(ab)=H(a)\cup H(b).$$

Denote T resp. $\overline{T}$ the set of all sets $H(z)$ resp. $\overline{H}(z)$ where $z\in F^+$.

Our results imply that $(T,\cup)$ and $(T,\cap)$ are semilattices.

Denote S_p the set of all elements (pa,pb) in $F^+\times F^+$ where $\varepsilon\neq p\in F^+$.

Theorem 1.3. S_p is a subsemigroup of the partial semigroup $P(F^+\times F^+,\otimes)$.

Proof. For arbitrary couple $(pa,pb),(pc,pd)$ of elements in S_p we have

$$(pa,pb)\otimes(pc,pd)=\left(pac(b\Delta c)^{-1},\ pbd(b\Delta c)^{-1}\right)\in S_p.$$

Thus it remains to show that

$$\begin{aligned}&[(pa,pb)\otimes(pc,pd)]\otimes(pr,ps)=\\&=(pa,pb)\otimes[(pc,pd)\otimes(pr,ps)]\end{aligned}$$

provided that $(pa,pb),(pc,pd),(pr,ps)\in S_p$.

To prove this equality, first we consider the following relations:

$$\text{(I.)}\quad [(pa,pb)\otimes(pc,pd)]\otimes(pr,ps) = $$
$$= \left(pac(b\Delta c)^{-1},\ pbd(b\Delta c)^{-1}\right)\otimes(pr,ps) =$$
$$= (pu,pv)\otimes(pr,ps) = \left(pur(r\Delta v)^{-1},\ psv(r\Delta v)^{-1}\right),$$

where $u = ac(b\Delta c)^{-1}$, $v = bd(b\Delta c)^{-1}$;

$$\text{(II.)}\quad (pa,pb)\otimes[(pc,pd)\otimes(pr,ps)] =$$
$$= (pa,pb)\otimes\left(pcr(r\Delta d)^{-1},\ psd(r\Delta d)^{-1}\right) =$$
$$= (pa,pb)\otimes(px,py) = \left(pax(x\Delta b)^{-1},\ pyb(x\Delta b)^{-1}\right),$$

where $x = cr(r\Delta d)^{-1}$, $y = sd(r\Delta d)^{-1}$.

Next we show that

$$pur(r\Delta v)^{-1} = pax(x\Delta b)^{-1}$$

and $pvs(r\Delta v)^{-1} = pby(x\Delta b)^{-1}$.

It is enough to show the following:

$$ur(x\Delta b) = ax(r\Delta v),$$
$$vs(x\Delta b) = by(r\Delta v).$$

These equalities can be written in the following equivalent form:

$$ac(b\Delta c)^{-1}r\left[(cr(r\Delta d)^{-1})\Delta b\right] =$$
$$= acr(r\Delta d)^{-1}\left[r\Delta(db(b\Delta c)^{-1})\right],$$
$$bd(b\Delta c)^{-1}s\left[(cr(r\Delta d)^{-1})\Delta b\right] =$$
$$= bsd(r\Delta d)^{-1}\left[r\Delta(db(b\Delta c)^{-1})\right],$$

whence it follows that it is enough to prove the relation

$$(b\Delta c)^{-1}\left[b\Delta(cr(r\Delta d)^{-1})\right] =$$
$$= (r\Delta d)^{-1}\cdot\left[r\Delta(db(b\Delta c)^{-1})\right].$$

Since

$$b = b_1(b\Delta c),\ c = c_1(b\Delta c),\ b_1\Delta c_1 = \varepsilon,$$
$$r = r_1(r\Delta d),\ d = d_1(r\Delta d),\ d_1\Delta r_1 = \varepsilon,$$

we obtain the following equivalent form of the above equality

$$(b\Delta c)^{-1}\left[\left((b\Delta c)b_1\right)\Delta\left(c_1r_1(b\Delta c)\,(r\Delta d)^{-1}\right)\right] =$$
$$= (r\Delta d)^{-1}\left[\left(r_1(r\Delta d)\right)\Delta\left(b_1d_1(b\Delta c)\,(r\Delta d)\,(b\Delta c)^{-1}\right)\right],$$

resp.

$$(b\Delta c)^{-1}\left[(b\Delta c)\left(b_1\Delta(c_1r_1)\right)\right] =$$
$$= (r\Delta d)^{-1}\left[(r\Delta d)\left(r_1\Delta(b_1d_1)\right)\right],$$

which is equivalent to the following

$$b_1\Delta(c_1r_1) = r_1\Delta(b_1d_1).$$

But this is true in view of $b_1\Delta c_1 = \varepsilon$, $r_1\Delta d_1 = \varepsilon$, and

$$b_1\Delta r_1 = r_1\Delta b_1.$$

2. Ont the ideals of $P(F^+ \times F^+, \otimes)$

Denote J_l the set of all elements (a,b) from $F^+ \times F^+$ for which $b \neq \varepsilon$. If $(a,b), (u,v) \in F^+ \times F^+$ and $b\Delta u \neq \varepsilon$, then

$$(a,b) \otimes (u,v) = \left(au(b\Delta u)^{-1}, bv(b\Delta u)^{-1}\right) \in J_l.$$

It can be seen immediately that $(x,y) \otimes (a,b) \in J_l$, if $(x,y) \in F^+ \times F^+$, $y\Delta a \neq \varepsilon$ and $(a,b) \in J_l$.

Therefore J_l is a partial left ideal of $P(F^+ \times F^+, \otimes)$.

Denote J_r the set of all elements (a,b) from $F^+ \times F^+$ for which $a \neq \varepsilon$.

If $(a,b), (u,v) \in F^+ \times F^+$ and $b\Delta u \neq \varepsilon$, then

$$(a,b) \otimes (u,v) = \left(au(b\Delta u)^{-1}, bv(b\Delta u)^{-1}\right) \in J_r.$$

It can be seen immediately that, $(a,b) \otimes (x,y) \in J_r$, if $(x,y) \in F^+ \times F^+$, $b\Delta x \neq \varepsilon$ and $(a,b) \in J_r$.

Therefore J_r is a partial right ideal of $P(F^+ \times F^+, \otimes)$.

It is easy to see, that

$$F^+ \times F^+ = J_l \cup \left(\bigcup_{y \in F^+ \setminus \varepsilon} (y,\varepsilon) \right) \cup (\varepsilon,\varepsilon),$$

and

$$F^+ \times F^+ = J_r \cup \left(\bigcup_{x \in F^+ \setminus \varepsilon} (\varepsilon,x) \right) \cup (\varepsilon,\varepsilon).$$

If $x \neq y$, $x\Delta y \neq \varepsilon$, $(x,y \in F^+)$ then $(\varepsilon,x) \otimes (y,\varepsilon) = (y(x\Delta y)^{-1}, x(x\Delta y)^{-1} \in J_l \cap J_r$, because in this case $y(x\Delta y)^{-1} \neq \varepsilon$ and $x(x\Delta y)^{-1} \neq \varepsilon$.

If $x \in F^+$, then $(\varepsilon,x) \otimes (x,\varepsilon) = (\varepsilon,\varepsilon)$.

Therefore

$$\left(\bigcup_{x \in F^+ \setminus \varepsilon} (\varepsilon,x) \right) \otimes \left(\bigcup_{y \in F^+ \setminus \varepsilon} (y,\varepsilon) \right) \subseteq (J_l \cap J_r) \cup (\varepsilon,\varepsilon).$$

Denote J_p, $(\varepsilon \neq p \in F^+)$ the set of all elements (pu,v), $(u,v \in F^+)$.

If $(x,y) \in F^+ \times F^+$ and $v\Delta x \neq \varepsilon$, then $(pu,v) \otimes (x,y) =$
$= \left(pux(v\Delta x)^{-1}, vy(v\Delta x)^{-1}\right)$,
where $pux(v\Delta x)^{-1} = pz$, $z \in F^+$.

Therefore J_p is a partial right ideal of $P(F^+ \times F^+, \otimes)$.

Denote ${}_pJ$, $(\varepsilon \neq p \in F^+)$ the set of all elements (u,pv), $(u,v \in F^+)$.

If $(x,y) \in F^+ \times F^+$ and $y\Delta u \neq \varepsilon$, then $(x,y) \otimes (u,pv) =$
$= \left(xu(y\Delta u)^{-1}, pyv(y\Delta u)^{-1}\right)$,
where $pyv(y\Delta u)^{-1} = pw$, $w \in F^+$.

Therefore ${}_pJ$ is a partial left ideal of $P(F^+ \times F^+, \otimes)$.

Theorem 1.3. proves that S_p is a subsemigroup of the partial semigroup $P(F^+ \times F^+, \otimes)$.

It is easy to see, that

$$J_p \cap {}_pJ = S_p.$$

We conclude that $J_p \cap {}_pJ$ is a subsemigroup of $P(F^+ \times F^+, \otimes)$.

Denote $\mathfrak{H}_r$, $(\varepsilon \neq p \in F^+)$ the set of all J_p.

It can be seen immediately that for the elements $J_p, J_q \in \mathfrak{H}_r$ we have

$$J_p \cap J_q = J_{pq(p\Delta q)^{-1}}.$$

Therefore $\mathfrak{L}_r = (\mathfrak{H}_r, \cap)$ is a semilattice.

Denote $\mathfrak{H}_l$, $(\varepsilon \neq p \in F^+)$ the set of all ${}_pJ$.

It can be seen immediately that for the elements ${}_pJ, {}_qJ \in \mathfrak{H}_l$ we have

$${}_pJ \cap {}_qJ = {}_{pq(p\Delta q)^{-1}}J.$$

Therefore $\mathfrak{L}_l = (\mathfrak{H}_l, \cap)$ is a semilattice.

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